ON HILL'S METHOD IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS. DETERMINATION OF THE CHARACTERISTIC EXPONENTS

(K METODU KHILLA V TEORII LINEINYKH DIFFERENTSIAL'NYKH URAVNENII C PERIODICHESKIMI KOEFFITSIENTAMI. OPREDELENIE KHARAKTERISTICHESKIKH POKAZATELEI)

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In [1], Equation (5.8) was given for the determination, in the first approximation, of the characteristic exponents in the case of parametric resonance of quasiharmonic quasistationary systems. In the present paper, equations are given which permit the determination of the characteristic exponents to an arbitrary degree of accuracy. These equations are employed in order to derive formulas, of importance in applications, which permit the determination, up to second-order terms inclusive, of the domain of parametric resonance of elastic systems. A simple stability criterion for solutions of second-order equations is also presented.

1. Consider the following system of differential equations:

$$\frac{d^2Y}{dt^2} + \mu N(\theta t)\frac{dY}{dt} + (C + \mu P(\theta t))Y = 0 \qquad (\theta > 0) \tag{1.1}$$

where μ is a small parameter, $C = (\omega_1{}^2, \ldots, \omega_m{}^2)$ is a diagonal matrix, and $\omega_j{}^2 > 0$ $(j = 1, \ldots, m)$, and

$$N(\tau) = \sum_{k=-l}^{l} N^{(k)} e^{ik\tau}, \qquad N^{(k)} = \| \nu_{js}^{(k)} \|_1^m \qquad (k = 0, \pm 1, \ldots, \pm l)$$

$$P(\tau) = \sum_{k=-l}^{l} P^{(k)} e^{ik\tau}, \qquad P^{(k)} = \| \pi_{js}^{(k)} \|_1^m \qquad (k = 0, \pm 1, \ldots, \pm l) \tag{1.2}$$

with complex numbers $\nu_{js}^{(k)}$, $\pi_{js}^{(k)}$.

Consider Hill's determinant [1] (see Formula (1.8)) for the system (1.1), which depends on the complex variable p. The matrix X of Hill's determinant consists of quasi-elements, i.e. of matrices of m rows and

m columns. On the principal diagonal one finds elements of the form [1] (putting $(k\theta)^2 \equiv 1$ for $k = 0$)

$$c_{rr}^{0k}(p) = -(k\theta)^{-2}((p + k\theta i)^2 + \mu\nu_{rr}^{(0)}(p + k\theta i) + \omega_r^2 + \mu\pi_{rr}^{(0)})$$
$$(r = 1, \ldots, m, \ k = 0, \pm 1, \ldots) \tag{1.3}$$

All the elements of the matrix X lying off the main diagonal have order μ. Let $\theta = \theta_0$, $\mu = 0$. Consider the set of $2m$ numbers

$$\omega_1, \ldots, \quad \omega_m, \quad -\omega_1, \ldots, \quad -\omega_m \tag{1.4}$$

and let us omit from this set those numbers which differ from a given one by terms of the form $k\theta_0$ (where k is an integer). Let us relabel these numbers $\rho_1, \ldots, \rho_n$ $(n \leqslant 2m)$. Let us denote the integer s by the symbol $[j]$ whenever either $\rho_j = \omega_s$ or $\rho_j = -\omega_s$, and set further

$$k_j \equiv (\rho_j - \rho_1)\theta_0^{-1} \tag{1.5}$$

From (1.3) it follows that Hill's determinant has, for $\mu = 0$, $\theta = \theta_0$, a zero of order n for $p = i\omega_g$, because in each row the diagonal element vanishes. These rows, as well as rows which are similarly placed in other infinite determinants, will be termed singular. Let us show that in this case the infinite Hill determinant may be reduced to a determinant may be reduced to a determinant of order n.

Let us divide the elements (1.3) of each row of the matrix X which contain $c_{rr}^{0k}(p)$ by the expression $-(k\theta)^{-2}((p + k\theta i)^2 + \omega_r^2)$ and denote the resultant matrix by U. Let us replace by zero all the elements in the singular rows (which contain the factor μ) of the matrix U and denote the resultant matrix by Z. In the matrix Z we have unity down the main diagonal in the singular rows and zeros otherwise. For $\mu = 0$ the matrices U, Z reduce to the identity matrix, and therefore the matrices U^{-1}, Z^{-1} may be expanded in power series in μ. The elements of the matrix Z^{-1} may be obtained from the corresponding elements of the matrix U^{-1} by disregarding all terms which become infinite for $p = i\omega_g$, $\theta = \theta_0$. As will be shown later, there always exists a positive number $\epsilon > 0$ such that whenever

$$|p - i\omega_g| \leqslant \varepsilon, \qquad |\theta - \theta_0| \leqslant \varepsilon, \qquad |\mu| \leqslant \varepsilon \tag{1.6}$$

then the matrix Z^{-1} exists and Det $Z^{-1} \neq 0, \neq \infty$. We also have

$$\operatorname{Det} X = \operatorname{Det}(XZ^{-1})\ (\operatorname{Det} Z^{-1})^{-1} \tag{1.7}$$

It is readily verified that the matrix XZ^{-1} has the property that along the non-singular rows the non-zero elements occur only along the main diagonal. Expanding Det (XZ^{-1}) in terms of these elements we obtain a determinant of order n whose elements are taken from the singular rows and from the columns which pass through the diagonal elements of the singular rows of the matrix XZ^{-1}.

The characteristic exponents $|p_j = i\omega_g| \leqslant \epsilon$, $j = 1, \ldots, n$, must satisfy the equation

$$D(p, \theta, \mu) \equiv \mathrm{Det}\, \| \delta_{sr} ((p + k_s\theta i)^2 + \rho_s^2) + \mu b_{sr}(p, \theta, \mu) \|_1^n = 0$$
$$(\delta_{ss} = 1, \ \delta_{sr} = 0, \ s \neq r) \tag{1.8}$$

$$b_{sr}(p, \theta, \mu) = \nu_{[s]\,[r]}^{(k_s - k_r)} (p + k_r\theta i) + \pi_{[s]\,[r]}^{(k_s - k_r)} - \mu \sum_{\chi,\,\alpha}' (\nu_{[s]\,\alpha}^{(k_s - \chi)} (p + \chi\theta i) + \pi_{[s]\,\alpha}^{(k_s - \chi)}) \times$$
$$\times \frac{\nu_{\alpha\,[r]}^{(\chi - k_r)} (p + k_r\theta i) + \pi_{\alpha\,[r]}^{(\chi - k_r)}}{(p + \chi\theta i)^2 + \rho_\alpha^2} + \mu^2 \sum_{\chi,\alpha,\beta,\gamma}' (\nu_{[s]\,\alpha}^{(k_s - \chi)} (p + \chi\theta i) + \pi_{[s]\,\alpha}^{(k_s - \chi)}) \times$$
$$\times \frac{\nu_{\alpha\beta}^{(\chi - \gamma)} (p + \gamma\theta i) + \pi_{\alpha\beta}^{(\chi - \gamma)}}{(p + \chi\theta i)^2 + \rho_\alpha^2} \, \frac{\nu_{\beta\,[r]}^{(\gamma - k_r)} (p + k_r\theta i) + \pi_{\beta\,[r]}^{(\gamma - k_r)}}{(p + \gamma\theta i)^2 + \rho_\beta^2} + \ldots \tag{1.9}$$

The mode of construction of the succeeding terms of the series (1.9) is obvious. The primes in the summations in (1.9) are to be interpreted to mean that the summations are to be extended over all possible different combinations of the integral values of the indices written under the summation signs, with the omission of those terms which, for $p = i\omega_g$, $\theta = \theta_0$, there is a zero in the denominator of the fraction involved.

The convergence of the series (1.9) follows from the convergence of the power series in μ for the matrix Z^{-1}, which converges provided that

$$|\mu| \sum_{k=-l}^{l} (|P^{(k)}| + (|p| + |k\theta|)\,|N^{(k)}|) \leqslant \min |(p + k\theta i)^2 + \omega_s^2| \tag{1.10}$$
$$(s = 1, \ldots, m, \ k = 0, \pm 1, \pm 2, \ldots \ \text{provided} \ \ \omega_s^2 - (\omega_g + k\theta_0)^2 \neq 0)$$

where $|P|$ denotes the norm of the matrix P.

If in Equations (1.8), (1.9) the elements $b_{sr}(p, \theta, \mu)$ are written up to terms of order $O(\mu^{k+1})$, then Equation (1.8) enables one to determine the characteristic exponents up to order $O(\mu^{k+1})$. In [1], in Equation (5.8), only the terms in Equation (1.9) up to order μ^0 were retained.

A similar procedure may be employed in order to discuss the resonance of quasistationary systems of first-order differential equations with periodic coefficients.

2. Consider the special case of the system (1.1) when $N(\tau) \equiv 0$ and $P(\tau)$ is a self-adjoint matrix [2]:

$$\frac{d^2Y}{dt^2}+(C+\mu P(\theta t))Y=0 \tag{2.1}$$

where C, $P(\tau)$ are the same as in (1.1) and (1.2). If in this case, in the expansion (1.9) for $b_{gr}(p,\theta,\mu)$, one retains only the terms of order μ^0, then Equation (1.8) will differ only by an unessential factor from the equation obtained by Iakubovich [3] in his consideration of dynamic stability.

Suppose that for a given g the equations

$$k\theta_0=\omega_g+\omega_h \qquad (g,\ h=1,\ldots,m,\ k=1,\ 2,\ 3\ldots) \tag{2.2}$$

hold only for a single choice of the numbers k, h. On the boundary of the domain of stability for a canonical system of differential equations there always exist multiple characteristic exponents. If, in (1.9), we retain only the terms of order μ, then, from the condition of the multiplicity of the roots of Equation (1.8) in the special case of the differential equation (2.1), it follows that

$$\begin{aligned}\theta_{\pm}=\frac{\omega_h+\omega_g}{k}+\frac{\mu}{2k}\Bigg\{&\frac{\pi_{gg}^{(0)}}{\omega_g}+\frac{\pi_{hh}^{(0)}}{\omega_h}\pm 2d-\frac{\mu}{4\omega_g}\left(\frac{\pi_{gg}^{(0)}}{\omega_g}\pm d\right)^2-\\&-\frac{\mu}{4\omega_h}\left(\frac{\pi_{hh}^{(0)}}{\omega_h}\pm d\right)^2-\frac{\mu}{\omega_g}\sum_{r=1}^{m}{\sum_{j=-l}^{l}}'\frac{\pi_{gr}^{(-j)}\pi_{rg}^{(j)}}{\omega_r^2-(\omega_g+j\theta_0)^2}-\\&-\frac{\mu}{\omega_h}\sum_{r=1}^{m}{\sum_{j=-l}^{l}}'\frac{\pi_{hr}^{(j)}\pi_{rh}^{(-j)}}{\omega_r^2-(\omega_h+j\theta_0)^2}\Bigg\}+O(\mu^3)\end{aligned} \tag{2.3}$$

$$d=\frac{1}{\sqrt{\omega_g\omega_h}}\left|\pi_{gh}^{(k)}-\mu\sum_{r=1}^{m}{\sum_{j=-l}^{l}}'\frac{\pi_{gr}^{(-j)}\pi_{rh}^{(k+j)}}{\omega_r^2-(\omega_g+j\theta_0)^2}\right| \tag{2.4}$$

Equation (2.3) is a generalization of Malkin's formula [2] for the determination of the domain of resonance for canonical systems. It is easy to write down the corresponding equations for higher-order approximations, but they are involved in appearance.

Example 1. Determine the domain of resonance for the canonical system of differential equations

$$\frac{d^2Y}{dt^2}+\left(\begin{pmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{pmatrix}+2\mu\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\cos\theta t\right)Y=0 \tag{2.5}$$

where

$$\theta_0=\omega_1+\omega_2, \qquad \theta_0=2\omega_1 \quad (\omega_1>\omega_2,\ \ \omega_1\neq k\omega_2,\ \ k=1,2,3,\ldots)$$

We have

$$\pi_{11}^{(1)} = \pi_{11}^{(-1)} = \alpha, \qquad \pi_{12}^{(1)} = \pi_{12}^{(-1)} = \pi_{21}^{(1)} = \pi_{21}^{(-1)} = \beta, \qquad \pi_{22}^{(1)} = \pi_{22}^{(-1)} = \gamma$$

and from Formulas (2.3) and (2.4) it follows that

$$\theta_{\pm} = \omega_1 + \omega_2 \pm \frac{\mu\beta}{\sqrt{\omega_1\omega_2}} - \frac{\mu^2\alpha^2}{\omega_1(\omega_1-\omega_2)(3\omega_1+\omega_2)} - \frac{\mu^2\gamma^2}{\omega_2(\omega_2-\omega_1)(3\omega_2+\omega_1)} - \\ - \frac{\mu^2\beta^2}{4\omega_1\omega_2(\omega_1+\omega_2)} + \ldots \tag{2.6}$$

$$\theta_{\pm} = 2\omega_1 \pm \frac{\mu\alpha}{\omega_0} - \frac{\mu^2\alpha^2}{8\omega_1^3} + \frac{\mu^2\beta^2}{\omega_1(9\omega_1^2-\omega_2^2)} + \frac{\mu^2\beta^2}{\omega_1(\omega_1^2-\omega_2^2)} + \ldots .$$

Equations (2.3) and (2.4) can be used for the determination of the domain of instability, up to order μ^2, for a single equation of type (2.1) (i.e. for Hill's equation). Since the characteristic exponents $p = 0.5\, k\theta i$ ($k = 0, \pm 1, \pm 2, \ldots$) lie on the boundary of the domain of simple parametric resonance [2, p. 341], we may employ, for the determination of the domain of simple parametric resonance, Equation (1.8):

$$D(0.5k\theta i,\ \theta,\ \mu) = 0 \quad (k = 0, \pm 1, \pm 2, \ldots)$$

3. Equation (1.8) may be employed to obtain (more simply than in [4, p. 598]) a criterion for the stability of solutions of a single second-order equation with periodic coefficients of the form

$$\frac{d^2y}{dt^2} + \mu f_1(\theta t, \mu)\frac{dy}{dt} + (\omega^2 + \mu f_2(\theta t, \mu))\, y = 0 \tag{3.1}$$

where $\omega^2 > 0$, the functions $f_1(\theta t, \mu)$, $f_2(\theta t, \mu)$ are real-valued functions with real arguments, and are continuous with respect to μ; here μ is a small parameter, $0 \leqslant \mu \leqslant \epsilon$, ($\epsilon > 0$):

$$f_1(\tau, \mu) = \sum_{k=-\infty}^{\infty} e^{ik\tau}\nu_k(\mu), \qquad |\nu_0(\mu)| + \sum_{k=-\infty}^{\infty} |k\nu_k(\mu)| \leqslant c_1 \\ f_2(\tau, \mu) = \sum_{k=-\infty}^{\infty} e^{ik\tau}\pi_k(\mu), \qquad \sum_{k=-\infty}^{\infty} |\pi_k(\mu)| \leqslant c_2 \tag{3.2}$$

On the boundary of the domain of instability in the plane of the parameters θ, μ, the characteristic exponents are numbers of the form $p = 0.5\, k\theta i$, that is, there exists a periodic or semi-periodic solution with period $2\pi\theta^{-1}$ (see [2, p. 316]).

Equation (1.8) (in this case the determinant is of second order) for the determination of the domain of instability for

$$\theta_0 = 2\omega k^{-1}, \qquad |\theta - \theta_0| \leqslant 2\omega(k+1)^{-1}k^{-1}, \qquad 0 \leqslant \mu \leqslant \varepsilon \qquad (k = 1, 2, \ldots) \tag{3.3}$$

is just

$$D(0.5k\theta i, \theta, \mu) \equiv |a_k(\theta, \mu)|^2 - |b_k(\theta, \mu)|^2 \tag{3.4}$$

From (1.9) we obtain (omitting for simplicity in writing, the arguments ν_k, π_k)

$$a_k(\theta, \mu) = \omega^2 - 0.25k^2\theta^2 + 0.5\mu k\theta\nu_0 i + \mu\pi_0 - \tag{3.5}$$
$$- \mu^2 \sum_{\chi}{}' \frac{(0.5k+\chi)\nu_{-\chi}\theta i + \pi_{-\chi}}{\omega^2 - (0.5k+\chi)^2\theta^2}(0.5\nu_\chi k\theta i + \pi_\chi) +$$
$$+ \mu^3 \sum_{\chi,\gamma}{}' \frac{(0.5k+\chi)\nu_{-\chi}\theta i + \pi_{-\chi}}{\omega^2 - (0.5k+\chi)^2\theta^2} \frac{(0.5k+\gamma)\nu_{\chi-\gamma}\theta i + \pi_{\chi-\gamma}}{\omega^2 - (0.5k+\gamma)^2\theta^2}(0.5k\nu_\gamma\theta i + \pi_\gamma) + \dots$$

and also

$$b_k(\theta, \mu) = -0.5\mu k\nu_k\theta i + \mu\pi_k - \mu^2 \sum_{\chi}{}' \frac{(0.5k+\chi)\nu_{-\chi}\theta i + \pi_{-\chi}}{\omega^2 - (0.5k+\chi)^2\theta^2} \times$$
$$\times (-0.5k\nu_{\chi+k}\theta i + \pi_{\chi+k}) + \mu^3 \sum_{\chi,\gamma}{}' \frac{(0.5k+\chi)\nu_{-\chi}\theta i + \pi_{-\chi}}{\omega^2 - (0.5k+\chi)^2\theta^2} \times$$
$$\times \frac{\nu_{\chi-\gamma}(0.5k+\gamma)\theta i + \pi_{\chi-\gamma}}{\omega^2 - (0.5k+\gamma)^2\theta^2}(-0.5k\nu_{\gamma+k}\theta i + \pi_{\gamma+k}) + \dots \tag{3.6}$$

The prime after the summation signs in (3.6) means that in the sums one is to omit the terms which have zero in the denominator for $\theta = 2\omega k^{-1}$, that is, that $\chi, \gamma, \dots, \neq 0, \neq k$. From (3.6) follows a rough estimate of the domain of convergence of the series (3.5), (3.6). Indeed, (3.3) must be satisfied, and also

$$|\mu| \leqslant \omega^2 k(2k-1)(c_1(k+1)(k+2)^2 + c_2(k+1)^2 k)^{-1} \qquad (k = 1, 2, \dots) \tag{3.7}$$

where c_1, c_2 are defined in (3.2). By the same method employed in [4, p. 599] the following theorem may be shown:

Theorem. Suppose that μ satisfies (3.7) and that θ is such that

$$2\omega(k+0.5)^{-1} \leqslant \theta \leqslant 2\omega(k-0.5)^{-1} \tag{3.8}$$

Solutions of Equation (3.1) are stable when:

(1)	$\mu\nu_0(\mu) > 0$,	$D(0.5k\theta i, \theta, \mu) > 0$	(asymptotically stable)
(2)	$\mu\nu_0(\mu) = 0$,	$D(0.5k\theta i, \theta\mu) > 0$	(bounded solutions)
(3)	$\mu\nu_0(\mu) > 0$,	$D(0.5k\theta i, \theta, \mu) = 0$	(there exists one periodic or semi-periodic solution with period $2\pi\theta^{-1}$)
(4)	$\mu\nu_0(\mu) = 0$,	$D(0.5k\theta i, \theta, \mu) = 0$, $b_k(\theta, \mu) = 0$	(there exist two linearly independent periodic or semi-periodic solutions)

In all other cases the solutions of (3.1) are unstable.

Taking into account the terms written in (3.5), (3.6), one may determine the domain of instability up to the order μ^3 inclusive.

Example 2. To determine a stability criterion for solutions of the equation

$$\frac{d^2y}{dt^2} + \mu(a + 2b\cos\theta t)\frac{dy}{dt} + \omega^2 y = 0 \qquad (a \geqslant 0) \tag{3.9}$$

when $\theta \approx \omega$, $\mu > 0$. We have $\nu_0 = a$, $\nu_{-1} = \nu_1 = b$, $k = 2$, $b(\theta, \mu) \equiv 0$. The stability condition is

$$(\omega^2 - \theta^2 + 2\mu^2 b^2\theta^2(\omega^2 - 4\theta^2)^{-1} + \ldots)^2 + (\mu a\theta - 4\mu^3 ab^2\theta^2(\omega^2 - 4\theta^2)^{-2} + \ldots)^2 \geqslant 0 \tag{3.10}$$

Hence the solutions of (3.9) are stable when

$$^4/_5\,\omega \leqslant \theta \leqslant {}^4/_3\,\omega, \qquad 0 \leqslant \mu \leqslant 0.125\omega^2(|a| + 2|b|^{-1} \qquad (a > 0)$$

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